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On the $\mathcal{F}$-abnormal maximal subgroups of finite groups[☆]

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Abstract

For any saturated formation $\mathcal{F}$ of finite groups containing all supersolvable groups, the groups in $\mathcal{F}$ are characterized by the $\mathcal{F}$-abnormal maximal subgroups.

MSC: 20D10; 20D30

1. Introduction

In this paper, G denotes a finite group. Let $\mathcal{F}$ be a saturated formation of finite groups and let $\mathcal{U}$ be the class of all finite supersolvable groups. $G^{\mathcal{F}}$ denotes the $\mathcal{F}$-residual of G, the intersection of all normal subgroups N of G satisfying $G/N \in \mathcal{F}$.

For a given saturated formation $\mathcal{F}$ of finite groups which contains $\mathcal{U}$, it is known that the groups G in $\mathcal{F}$ can be characterized by the minimal subgroups of $G^{\mathcal{F}}$ and by the maximal subgroups of the Sylow subgroups of $G^{\mathcal{F}}$ (see for example [1–3]). In this paper, we shall continue this topic by considering the $\mathcal{F}$-abnormal maximal subgroups of the groups.

A maximal subgroup M of G is called $\mathcal{F}$-normal in G if $G/M_G \in \mathcal{F}$; otherwise M is said to be $\mathcal{F}$-abnormal in G (see [4,5]), where M_G denotes the core of M in G, the largest normal subgroup of G which is contained in M. It is clear that G belongs to $\mathcal{F}$ if and only if all the maximal subgroups of G are $\mathcal{F}$-normal in G. In this paper, we will characterize the groups in $\mathcal{F}$ by considering the $\mathcal{F}$-abnormal maximal subgroups of the groups.

2. Main results

Let $\mathcal{M}(G)$ denote the set of $\mathcal{F}$-abnormal maximal subgroups of G. Write $b(\mathcal{F})$ for the Q-boundary of $\mathcal{F}$, i.e., $G \in b(\mathcal{F})$ if and only if $G \notin \mathcal{F}$ while $G/N \in \mathcal{F}$ for any $1 < N \lhd G$.

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Lemma 1. *A maximal subgroup M of G is $\mathcal{F}$-abnormal in G if and only if $G = G^{\mathcal{F}}M$.*

Proof. If $G/M_G \in \mathcal{F}$, then $G^{\mathcal{F}} \subseteq M_G \subseteq M$. Conversely, if $G^{\mathcal{F}} \subseteq M$, then $G^{\mathcal{F}} \subseteq M_G$. It follows that G/M_G is a homomorphic image of $G/G^{\mathcal{F}}$; hence G/M_G belongs to $\mathcal{F}$. $\square$

Definition 2 (*Yaoqing Zhao [6, Definition]*). Given a maximal subgroup M of a group G, a θ-completion of M in G is any subgroup C such that C is not contained in M while M_G is contained in C and C/M_G has no proper normal subgroup of G/M_G.

Denote by $\theta I(M)$ the set of all θ-completions of M in G. A θ-completion C is said to be maximal in $\theta I(M)$ if there is no θ-completion D in $\theta I(M)$ such that $C < D$. We call C a maximal θ-completion.

By making use of some information on the maximal θ-completions, Zhao Yaoqing [6] obtain many results which imply a group to be solvable, supersolvable or nilpotent. However, we notice that the further research in the past relied on the maximality condition of θ-completions. Therefore, it should be interesting to weaken or dispense with the maximality imposed on θ-completions. Now we give the following definition of s–θ-completion for the first time.

Definition 3. Given a maximal subgroup M of a group G, a θ-completion C of M is called an s–θ-completion if either $C = G$ or there exists a subgroup B of G such that

(i) C is a maximal subgroup of B;
(ii) B is not a θ-completion of M.

The next lemma is very important for s–θ-completions.

Lemma 4. *Suppose that M is a maximal subgroup of G and C is an s–θ-completion of M. If C is not normal in G and G/M_G has a unique minimal normal subgroup K/M_G, then*

(i) C is a maximal subgroup of the group CK.
(ii) C is not a subgroup of K.

Proof. (i) Since C is not normal in G, we have $C < G$ and $C \neq K$. Because C is an s–θ-completion of M, by Definition 3, we can see that there exists a subgroup E of G such that C is a maximal subgroup of E and E is not a θ-completion of M. Therefore, E/M_G contains a chief factor B/M_G of G with $B < E$. By hypothesis, K/M_G is the unique minimal subgroup of G/M_G. So $B = K$ and $CK \leq E$. Because C is a maximal subgroup of E, we have $C = CK$ or $CK = E$. If $C = CK$, then $K \leq C$. But C is not normal in G, and hence $K < C$; this is in contradiction with the fact that C is a θ-completion of M. Therefore $CK = E$ and C is a maximal subgroup of CK.
(ii) If $C \leqslant K$, then $E = CK = K = B$. But $B < E$, a contradiction. Hence $C \nleq K$ holds. $\square$

Lemma 5 (*Shirong Li [7, Lemma 2]*). *If a finite group G is factorizable in the form $G = CD$, where neither $|C|$ nor $|D|$ is divisible by 4, then G is 2-nilpotent. In particular, G is solvable.*

Lemma 6 (*Bhattacharya and Mukherjee [8, Theorem 3]*). *For any finite group G, the intersection of maximal subgroups of G of composite index is supersolvable.*

Huppet's theorem on the supersolvable groups says that a finite group G is supersolvable if and only if every maximal subgroup of G is of index a prime. Now, we give a natural generalization by means of saturated formations containing all supersolvable groups.

Lemma 7. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then the finite group $G \in \mathcal{F}$ if and only if every $\mathcal{F}$-abnormal maximal subgroup of G has index a prime.*

Proof. If $G \in \mathcal{F}$, then all maximal subgroups of G are $\mathcal{F}$-normal, so the condition is vacuously satisfied.

Conversely, assume that G is a counterexample satisfying the condition. If $G^{\mathcal{F}} = G$, then all maximal subgroups of G are $\mathcal{F}$-abnormal in G. Hence each maximal subgroup of G has index a prime by hypothesis, and G is supersolvable. In particular, $G \in \mathcal{F}$, a contradiction. Let $G^{\mathcal{F}} < G$. Then $G^{\mathcal{F}}$ is contained in each $\mathcal{F}$-normal maximal subgroup of G, and hence in each maximal subgroup of G of composite index. By Lemma 6, $G^{\mathcal{F}}$ is supersolvable, and hence solvable. Because $G^{\mathcal{F}} > 1$, let $G^{\mathcal{F}}/N$ be a chief factor of G. Then $G^{\mathcal{F}}/N$ is a group of prime power order and G/N satisfies the condition. So $N = 1$. That is, $G^{\mathcal{F}}$ is an abelian minimal normal subgroup of G. Also, we have $G^{\mathcal{F}} \nsubseteq \Phi(G)$, so there exists a maximal subgroup M of G such that $G^{\mathcal{F}} \nsubseteq M$. By Lemma 1, $M \in \mathcal{M}(G)$. By hypothesis, $|G : M| = p$, a prime number. It follows that $G^{\mathcal{F}}$ is cyclic of order p. Then we have $G \in \mathcal{F}$. This is a final contradiction. The lemma is now proved. □

Another form of Lemma 7 is the following corollary:

Corollary 8. *Let $\mathcal{F}$ be as in Lemma 7. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of G of composite index is $\mathcal{F}$-normal in G.*

Let $\mathcal{M}_c(G) = \{M : M \in \mathcal{M}(G)$ and $|G : M|$ a composite$\}$; we have

Theorem 9. *Let $\mathcal{F}$ be a saturated formation which contains $\mathcal{U}$. Then the finite group G belongs to $\mathcal{F}$ if and only if, for every M in $\mathcal{M}_c(G)$, M has an s–θ-completion C such that $G = CM$ and C/M_G has square free order.*

Proof. If $G \in \mathcal{F}$, then $\mathcal{M}(G) = \phi$, so the conclusion holds.

Conversely, let G be a finite group satisfying the condition. Assume that G is a counterexample, that is, G does not belong to $\mathcal{F}$. We work for a contradiction.

Because $\mathcal{F}$ is a saturated formation, there exists a normal subgroup N of G such that $G/N \in b(\mathcal{F})$. Then G/N possesses a unique minimal normal subgroup U/N and $G/U \in \mathcal{F}$. We have $U/N = (G/N)^{\mathcal{F}} = G^{\mathcal{F}}N/N$.

(1) There exists an $M \in \mathcal{M}_c(G)$ such that $N \leq M$.

If $\mathcal{M}_c(G/N) = \phi$, then $G/N \in \mathcal{F}$ by Lemma 7, contrary to $G/N \notin \mathcal{F}$. So $\mathcal{M}_c(G/N) \neq \phi$, that is, there exists a maximal subgroup M of G of composite index such that $N \leq M$ while $U \nsubseteq M$. In particular, $G^{\mathcal{F}} \nsubseteq M$. So we have $M \in \mathcal{M}_c(G)$ by Lemma 1, as desired.

(2) U/N is a non-solvable characteristic simple group.

Take M as in (1). By hypothesis, M has an s–θ-completion C such that $G = CM$ and C/M_G has square free order. If U/N is solvable, then $|U/N|$ is a power of a prime p and $|G : M| \mid |U/N|$. That is, M has as index a power of p. On the other hand, from $|G : M| = |C : C \cap M| \mid |C : M_G|$, we have that $|G : M|$ must be square free. Thus $|G : M|$ must be a prime, contrary to $|G : M|$ being composite.

(3) $N = M_G$, C is maximal in CU and $G^{\mathcal{F}} \nsubseteq C$.

From (1), we can see that $N \leq M$, so $N \leq M_G$. If $N < M_G$, then $G/M_G \in \mathcal{F}$, a contradiction. Thus $N = M_G$. Because U/N is non-solvable, while C/N is of square free order and hence solvable, we have that $C \ntrianglelefteq G$. We thus can apply Lemma 4 to see that C is maximal in CU. If $G^{\mathcal{F}} \subseteq C$, then $U = G^{\mathcal{F}}N \leq C$. This is impossible because C/N is solvable while U/N is non-solvable. This contradiction shows that $G^{\mathcal{F}} \nsubseteq C$.

(4) Every $M \in \mathcal{M}(G)$ has square free index in G.

If $|G : M|$ is composite, then $M \in \mathcal{M}_c(G)$. By hypothesis, M has an s–θ-completion C such that $G = CM$ and C/M_G has square free order. Since $|G : M| = |C : M \cap C| \mid |C : M_G|$, it follows that $|G : M|$ is square free.

(5) C is maximal in G.

By (3), C/N is maximal in CU/N and $C/N = C/M_G$ has square free order and is hence solvable. Because CU/N is non-solvable, we have $C/N \cap U/N \neq 1$. Let p be the largest prime factor dividing $|C/N \cap U/N|$, P/N a Sylow p-subgroup of $(C \cap U)/N$. Then $|P/N| = p$ and P/N is normal in C/N, but not normal in U/N. Because C/N is maximal in CU/N and U/N contains no non-trivial normal subgroup of prime power order, we see that P/N must be a full Sylow p-subgroup of U/N. By the Frattini argument, $G/N = N_{G/N}(P/N)U/N = (H/N)(U/N)$, where H/N is a maximal subgroup that contains $N_{G/N}(P/N)$ and hence C/N. We now have $G = HU$, $N < C \leq H < G$. If $C = H$, then the claim is complete.

Suppose that $C < H$. We can assert $C \cap U = H \cap U$.

In fact, if $C \cap U < H \cap U$, then $C < C(H \cap U) = CU \cap H \leq CU$, which forces $CU = CU \cap H$ since C is maximal in CU. Consequently $U \leq H$, a contradiction. This proves our assertion.

Now, as $G = HU$, we have $|G : H| = |U : U \cap H| = |U : C \cap U|$. Since it is easy to see that $N = H_G$, we have $G/H_G = G/N \notin \mathcal{F}$, that is $H \in \mathcal{M}(G)$. By (4), it follows that $|G : H|$ is square free. Because $|G : H| = |U : C \cap U|$, we can see that $|U/N|$ is cube free and hence U/N is a non-abelian simple group with Sylow 2-subgroups of order 4. Such non-abelian simple groups have been classified by the Gorenstein–Walter theorem (see [9]); only these groups are $PSL(2,q)$, $q \equiv 3, 5 (\mathrm{mod}\ 8)$.

Let $q = r^f$ for some prime τ. Because $r^f \equiv 3, 5 (\mathrm{mod}\ 8)$, $f = 1$ or 2. It is well known that the automorphism group of $PSL(2, r^f)$ is a semi-direct product of $PSL(2, r^f)$ by Z_f, the cyclic group of order f and $|PGL(2, r^f)/PSL(2, r^f)| = 2$.

Consider the case when $f = 2$. Since $PSL(2, 4) \cong PSL(2, 5)$, we may let r be odd. By the table of subgroups of $PSL(2, q)$ (see [10]), we know that $U/N = PSL(2, r^2)$ has at least one maximal subgroup D/N which is a non-nilpotent dihedral subgroup of order $r^2 - 1$ or $r^2 + 1$. Without loss of generality, assume that $|D/N| = r^2 - 1$. Let s be an odd prime divisor of $r^2 - 1$. Since $|PSL(2, r^2)| = r^2(r^2 - 1)(r^2 + 1)/2$, we can choose a Sylow s-subgroup S/N of U/N such that $S/N \subseteq D/N$. Then $D/N \leq N_{G/N}(S/N) \leq K/N$, where K/N is a maximal subgroup of G/N and $G = KU$. We have $D/N \leq (K \cap U)/N$ and so $K \cap U = D$ since D/N is maximal in U/N. Therefore $|G : K| = |U : K \cap U| = |U : D| = r^2(r^2 + 1)/2$. But it is easy to see that $K \in \mathcal{M}(G)$, so $|G : K|$ is square free by (4), a contradiction. Thus we deduce that $f = 1$, namely $U/N \cong PSL(2, r)$.

Now, as U/N is the unique minimal subgroup of G/N, we have $PSL(2, r) \leq G/N \leq PGL(2, r) = \mathrm{Aut}(PSL(2, r))$. If $CU = G$, then C is maximal in G. If $CU < G$, we have $CU = U$, so $C \leq U$. But $C \ntrianglelefteq G$; then $C \nleq U$ by Lemma 4, a contradiction. Now the claim is proved.

(6) Final contradiction.

If $|G : C|$ is a prime, as C is maximal in G and C/N is of square free order, G/N would be solvable, a contradiction. So $|G : C|$ is composite. Thus, by (3) and (5), we have $C \in \mathcal{M}_c(G)$. By hypothesis, we can see that C has an s–θ-completion D such that $G = CD$ and D/C_G has square free order. It is obvious that $N = C_G$. Now, $G/N = C/N \cdot D/N$, and both C/N and D/N have square free order. By Lemma 5, G/N would be solvable, which is a final contradiction. The proof is now complete. □

Theorem 10. *Let $\mathcal{F}$ be a saturated formation which contains $\mathcal{U}$ and let G be finite group which is S_4-free. Then G belongs to $\mathcal{F}$ if and only if every member M of $\mathcal{M}_c(G)$ has an s–θ-completion C such that C/M_G is cyclic with $|C/M_G| \geq |G : M|$.*

Proof. If $G \in \mathcal{F}$, then $\mathcal{M}(G) = \phi$, so the conclusion holds.

Conversely, let G be a counterexample. Then there exists a normal subgroup N of G such that $G/N \in b(\mathcal{F})$. Let U/N be the unique minimal normal subgroup of G/N. As in the proof of Theorem 9, we have

(1) $U/N = (G/N)^{\mathcal{F}} = G^{\mathcal{F}}N/N$ and $\mathcal{M}_c(G/N) \neq \phi$.

In the next argument, we want to show that $G/N \cong S_4$, which is contrary to the hypothesis, and so the proof is complete.

(2) U/N is an elementary abelian p-group for some prime p.

Assume that U/N is non-solvable. Then U/N is a non-abelian characteristic simple group. In particular, U/N has no non-trivial normal subgroup of prime power order. As in the proof of claim (1) of Theorem 9, it is clear that there exists $M \in \mathcal{M}_c(G)$ such that $G = MU = MG^{\mathcal{F}}$. By hypothesis, M possesses an s–θ-completion C such that C/M_G is cyclic. It is easy to see that $N = M_G$. By Lemma 4, we can see that C/N is maximal in CU/N. Because any finite group containing an abelian maximal subgroup is solvable, CU/N, and hence U/N, would be solvable, contrary to the assumption.

(3) $|G : M| = 4$.

By (2), $|U/N| = p^n$, $n \geq 1$, and we have $|G : M| = |G/N : M/N| = |U/N|$. We only need to show that $|U : N| = 4$.

Let C be an s–θ-completion of M such that C/M_G is cyclic and $|C/M_G| \geq |G : M|$. Then $N = M_G$ and C is maximal in CU by Lemma 4. Put $E = CU$. We claim $C \lhd E$.

Otherwise E/N is not a p-group and hence C/N is not a p-group. In particular, we have $|C/N| \neq |U/N|$. It follows by hypothesis that $|C/M_G| \geq |G : M|$; we deduce $|C| > |U|$. Let B be a conjugate of C in E such that $B \neq C$. Then $|B| > |U|$ and $|B||C|/|B \cap C| = |BC| \leq |E| = |U||C|/|U \cap C|$, and so $|B \cap C| > |U \cap C|$. This implies

$B \cap C \nsubseteq U$. On the other hand, U/N is the unique minimal normal subgroup of G/N and abelian, so $C_G(U/N) = U$. Thus we see that $B \cap C$ does not centralize U/N. Let $X/N = (B \cap C)/N$. Then $|X/N| > |(U \cap C)/N|$, and both B and C centralize X/N because any B/N or C/N is cyclic. It follows that $E = \langle B, C\rangle$ centralizes X/N. In particular, U centralizes X/N, or equivalently X centralizes U/N, which is a contradiction and thus $C \lhd E$ is proved.

Now C is maximal and normal in E, so $|E : C|$ must be a prime and C/N must be a p-group (otherwise $C_G(U/N) > U$). Write $T = U \cap C$. Then $|U : T|$ is a power of p. Since $|U : T| = |E : C|$, we have $|U : T| = p$. Also, T/N is a cyclic subgroup of U/N, so $|T/N| = p$. We thus deduce $|U/N| = p^2$. What remains is to show $p = 2$.

By the above, we have $|C| \geq |U| > |T|$, and it follows that $E > U$. Let V be a subgroup of E containing U such that $|V : U| = p$. Then $V \cap C > T$ and $(V \cap C)/N$ is cyclic. Hence V/N is a group of order p^3 with exponent p^2. Let $Q = V \cap M$. Then Q/N is a subgroup of order p of V/N and $Q/N \nsubseteq U/N$, and hence, V/N possesses more than p^2 elements of order dividing p. If $p > 2$, the only p-groups of order p^3 having this property have exponent p, which is a contradiction. Thus we get $p = 2$.

(4) The final contradiction.

By the above, there exists a maximal subgroup $M \in \mathcal{M}_c(G)$ having index 4 and $N \leq M$. Considering the permutation representation of G/N on four cosets of M/N, we see that G/N is isomorphic to a subgroup of S_4. Because G/N is not in $\mathcal{F}$, in particular, G/N is non-supersolvable, we can conclude $G/N \cong S_4$, contrary to the hypothesis. $\square$

Corollary 11. *Let $\mathcal{F}$ be a saturated formation which contains $\mathcal{U}$ and let G be a finite group which is S_4-free. Then $G \in \mathcal{F}$ if and only if every member M of $\mathcal{M}_c(G)$ possesses a maximal completion C such that $G = CM$ and C/M_G is cyclic.*

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